Hamiltonian symmetries of the gauge theories in the light front

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[arXiv: 2302.xxxxx]

HOLOGRAPHYCL

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Outline

1. Infrared structure of gauge theories

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2. Hamiltonian analysis of electromagnetism in the null foliation

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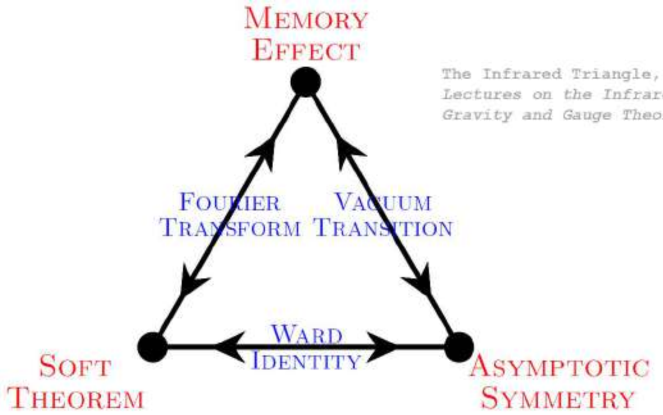
3. New symmetry generator; Beyond $\mathrm{U}(1)$
4. Extension to Yang-Mills theory
5. Discussion

Infrared structure of gauge theories

IR region of theories with massless particles in asymptotically flat spaces



The Infrared Triangle, A. Strominger, *Lectures on the Infrared Structure of Gravity and Gauge Theory* arXiv:1703.05448

Infrared structure of gauge theories

Motivation

- **Hamiltonian treatment of asymptotic symmetries**

  [Bondi, van der Burg, Metzner 1962; Sachs 1962]

  BMS symmetry – infinite-dimensional asymptotic symmetry at the null boundary of 4D asymptotically flat spacetimes

- **Celestial holography**

  Duality between massless 4D asymptotically flat spacetimes and 2D CFT on the celestial sphere

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- **Asymptotic symmetries in electromagnetism and Yang-Mills theory**

- 2D realization of soft symmetries in electromagnetism

  [He, Mitra, Porfyriadis, Strominger 2014; Nande, Pate, Strominger 2018]

- Extension to Yang-Mills theory

  [Strominger 2014; He, Mitra, Strominger, 2016]

Infrared structure of gauge theories

Vacuum degeneracy in gauge theories ($\omega \longrightarrow 0$)

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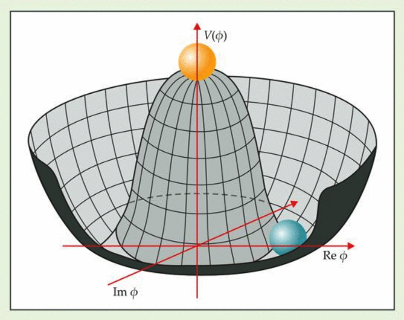

Spontaneous symmetry breaking, J. Lykken, M. Spiropulu, The future of the Higgs boson, Physics Today 66, 12, 28 (2013)

$\Rightarrow$ **Interest in boundary dynamics of light and massless particles**

Infrared structure of gauge theories

What to expect at the null infinity?

Infrared structure of gauge theories

What to expect at the null infinity?

- Realisation of a canonical analysis in a null foliation [Dirac 1949]
- The induced metric on a null hypersurface is degenerate
- Double-null foliation $= 2+2$ formalism in GR, complicated symplectic structure, difficult to quantize [d'Inverno, Smallwood 1980]
- Ashtekar variables in GR, simpler symplectic structure, but still difficult to quantize [Ashtekar 1986, 1987]

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Peculiarities of the light-front dynamics in the Minkowski space

- **Light-cone coordinates** $x^{\pm} = \frac{x^0 \pm x^3}{\sqrt{2}}$; Time coordinate $u = x^-$
- **Increased number of isometries** of the surface $u = \text{const.}$ compared to $t = x^0 = \text{const.}$ (one more because of degenerated direction)

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- **Dispersion equation** for a massive scalar

  $p^2 = m^2 \quad \Rightarrow$ Energy $E = p^- = \frac{(p^{\perp})^2 + m^2}{2p^+}$

$\Rightarrow$ Consequences: $p^+ > 0$ and trivial physical vacuum, $p^{\mu}_{\mathrm{vac}} = 0$

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- **Boundary conditions** in the light front formalism

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- **Light-cone actions** are first order in velocities [Steinhardt 1980]

  Kinetic term $T = -\frac{1}{2}\,(\partial\phi)^2 = \dot{\phi}\,\partial_+\phi - (\nabla_\perp\phi)^2$

  $\Rightarrow$ The canonical momentum $\pi = \partial_+\phi$ is not invertible

★ New constraint $\chi \equiv \pi - \partial_+\phi \approx 0$

★ It does not commute with itself, $\{\chi(x), \chi(x')\}_{u=u'} = -2\partial_+\delta(x-x')$

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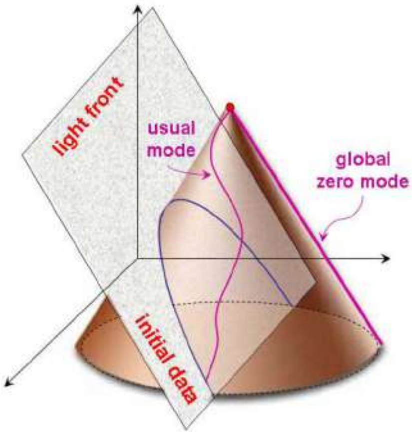
- **Global zero mode** in massless theories [Alexandrov, Speziale 2015]

- A massless particle worldline is parallel to the light front hypersurface, not determined by the initial data
- It has vanishing energy, $E = P^- \to 0$ (Fourier momenta $P^- = 0$, $P^\perp = 0$)

Global zero mode



Global zero mode, *First order gravity on the light front*, S. Alexandrov, S. Speziale, Phys.Rev.D 91 (2015) 6, 064043

Hamiltonian analysis of electromagnetism in the null foliation

Null foliated reference frame

- **Minkowski metric in $D = 4$ in the spherical coordinates** (t, r, y^A)

$$M_4 : \quad \mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2$$

$$\mathbb{S}^2 : \quad \mathrm{d}\Omega^2 = \gamma_{AB}(y)\,\mathrm{d}y^A \mathrm{d}y^B$$

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- **Time coordinate** $\quad u = t - \epsilon r\,, \quad -1 \leq \epsilon \leq 1$

$\epsilon = 1$	retarded time
$\epsilon = 0$	proper time of a massive particle
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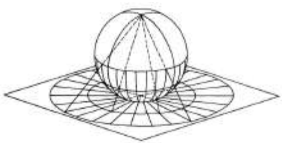
Stereographic projection, T. Apostol, Mathematical Analysis (1973)



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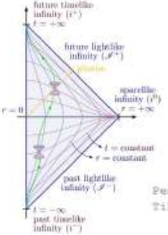
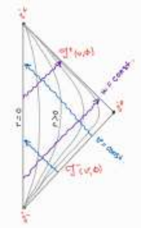
- **Coordinates on** $\mathbb{S}^2$: **stereographic projection** $(\theta, \varphi) \to y^A = (z, \bar{z})$

$$z = \mathrm{e}^{\mathrm{i}\varphi}\cot\frac{\theta}{2}\,, \quad \bar{z} = \mathrm{e}^{-\mathrm{i}\varphi}\cot\frac{\theta}{2}$$

Hamiltonian analysis of electromagnetism in the null foliation



Penrose diagram of the Minkowski space, TikZ.net



- **Minkowski metric $\mathfrak{g}_{\mu\nu}$ in the coordinates $x^\mu = (u, r, y^A)$**:

$$\boxed{ds^2 = -du^2 - 2\epsilon\, du dr + \left(1-\epsilon^2\right) dr^2 + r^2 d\Omega^2} \qquad \sqrt{\mathfrak{g}} = r^2\sqrt{\gamma}$$

- **$\mathbb{S}^2$ metric in the complex coordinates**

$$\gamma_{AB} = \begin{pmatrix} 0 & \gamma_{z\bar{z}} \\ \gamma_{z\bar{z}} & 0 \end{pmatrix}, \quad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2} = \sqrt{\gamma}$$

Hamiltonian analysis of electromagnetism in the null foliation

- **Electromagnetic action in the background $\mathfrak{g}_{\mu\nu}$**

$$I[A] = -\frac{1}{4e^2}\int \mathrm{d}^4x\,\sqrt{\mathfrak{g}}\,F^{\mu\nu}F_{\mu\nu} \qquad (F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu)$$

- **Canonical momenta** $\pi^\mu = -\frac{1}{e^2}\sqrt{\mathfrak{g}}\,F^{u\mu}$

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In components:

$$\boxed{\begin{array}{llll} \pi^u & = 0 & \dot{A}_u & \times \\ \pi^r & = \frac{r^2}{e^2}\sqrt{\gamma}\,F_{ur} & \dot{A}_r & \surd \\ \pi^A & = -\frac{1}{e^2}\sqrt{\gamma}\gamma^{AB}\left[(\epsilon^2-1)F_{uB} - \epsilon F_{rB}\right] & \dot{A}_B & ? \end{array}}$$

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- The limit $\epsilon^2 \to 1$ is discontinuous

- The action in the light-cone ($\epsilon^2 = 1$) has an additional constraint

Hamiltonian analysis of electromagnetism in the null foliation

In the Bondi reference frame $(\epsilon^2 = 1)$

- **Primary constraints** $\boxed{\pi^u \approx 0\,, \quad \chi^A \equiv \epsilon\pi^A - \frac{1}{e^2}\sqrt{\gamma}\gamma^{AB}F_{rB} \approx 0}$

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- **Total Hamiltonian** [Dirac 1964]

$$\mathcal{H}_T = \frac{e^2(\pi^r)^2}{2r^2\sqrt{\gamma}} + \frac{e^2\tilde{\pi}_A\pi^A}{2\sqrt{\gamma}} + \frac{\sqrt{\gamma}}{4e^2r^2}\tilde{F}^{AB}F_{AB} - A_u\partial_i\pi^i + \lambda_u\pi^u + \lambda_A\chi^A$$

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- **Canonical Poisson brackets** $\{A_\mu(x), \pi^\nu(x')\}_{u=u'} = \delta^\nu_\mu\,\delta^{(3)}(x-x')$
- **Evolution** $\dot{\Phi}(x) = \{\Phi(x), H_T\}$

Hamiltonian analysis of electromagnetism in the null foliation

Symplectic matrix

$$\boxed{\left\{\chi^A(x), \chi^B(x')\right\} = \Omega^{AB}(x, x') \equiv -\frac{2\epsilon}{e^2}\sqrt{\gamma}\gamma^{AB}\partial_r\delta^{(3)}}$$

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- If Ω^{AB} is invertible: χ^A are **second class** (eliminate redundant fields)

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One possibility

- Ω^{AB} is invertible because γ^{AB} is invertible $\Rightarrow \chi^A$ are second class
- Reduced phase space $\chi^A = 0$ [Goldberg 1991, Majumdar 2022]

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Second possibility

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- Ω^{AB} is infinite-dimensional matrix and it has zero modes

$$\int \mathrm{d}^3x'\,\Omega^{AB}V'_B = -\frac{2\epsilon}{e^2}\sqrt{\gamma}\gamma^{AB}\partial_r V_B = 0 \quad \Rightarrow \quad V_B = V_B(y)$$

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- Ω^{AB} is infinite-dimensional matrix and it has zero modes

$\Rightarrow \chi^A_{(0)}(y)$ **is first class constraint** (r-independent part of the constraint)

Hamiltonian analysis of electromagnetism in the null foliation

Consistency conditions

Hamiltonian analysis of electromagnetism in the null foliation

Consistency conditions

- **Conservation of constraints during their evolution**

$$\dot{\pi}^u = 0 \qquad \Rightarrow \quad \chi = \partial_i \pi^i \approx 0 \quad \text{(differential Gauss law)}$$

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- **The multiplier λ_A is not fully determined**

$$\partial_r \lambda_A = -\frac{\epsilon e^2}{2\sqrt{\gamma}} \partial_r \tilde{\pi}_A - \frac{1}{2r^2} \nabla^B F_{AB} + \frac{\epsilon e^2}{2r^2} \partial_B \left(\frac{\pi^r}{\sqrt{\gamma}} \right)$$

$$\lambda_A = \bar{\lambda}_A + \Lambda_A(y)$$

- A free function $\Lambda_A(y)$ is due to the zero modes of Ω^{AB}
- $\bar{\lambda}_A$ – determined part of λ_A

Hamiltonian analysis of electromagnetism in the null foliation

Summary of the constraints

$$\boxed{\begin{array}{ll} \text{Primary constraints:} & \pi^u\,, \quad \chi^A = \epsilon\pi^A - \frac{1}{e^2}\sqrt{\gamma}\gamma^{AB}F_{rB} \\ \text{Secondary constraint:} & \chi = \partial_i\pi^i\,. \end{array}}$$

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Nature of the constraints

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Primary constraints:	$\pi^u\,, \quad \chi^A = \epsilon\pi^A - \frac{1}{e^2}\sqrt{\gamma}\gamma^{AB}F_{rB}$
Secondary constraint:	$\chi = \partial_i \pi^i\,.$

Nature of the constraints

- π^u – first class, A_u is a multiplier in the Hamiltonian
- χ – first class, differential Gauss law, $\pi^i = \sqrt{\gamma}\,E^i$
- $\chi^A_{(0)}$ – first class, r-independent part of the constraint
- $\chi^A_{(n)}$ $(n \geq 1)$ – second class, coefficients of the Taylor expansion in $1/r$
- We have to expand all the fields asymptotically in the vicinity of the boundary $r = const \to \infty$.

Hamiltonian analysis of electromagnetism in the null foliation

Standard asymptotic conditions of the fields [Strominger 2014]

$$\boxed{\begin{aligned} A_u &= \mathcal{O}(\tfrac{1}{r})\,, & A_r &= \mathcal{O}(\tfrac{1}{r^2})\,, & A_A &= \mathcal{O}(r^0)\,, \\ \pi^u &= 0\,, & \pi^r &= \mathcal{O}\left(r^0\right)\,, & \pi^A &= \mathcal{O}(\tfrac{1}{r^2}) \end{aligned}}$$

- **Boundary fields**: $A_{(0)A}, \quad \pi^r_{(0)}$

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Summary

1^{st} class constraints	Parameters	Generators	Charges
$\pi^u,\ \chi = \partial_i \pi^i$	$\theta_u,\ \theta$	$G[\theta]$	$Q[\theta]$
$\chi^A_{(0)}$	$\eta_A(y)$	$S[\eta]$	$Q_s[\eta]$

- **Smeared generators**

$$G[\theta] = \int \mathrm{d}^3x\,\left(\theta\,\partial_i \pi^i + \theta_u \pi^u\right) \quad \text{standard U(1) symmetry}$$

$$S[\eta] = \int \mathrm{d}^3x\,\eta_A \chi^A \quad \text{asymptotic symmetry}$$

A new symmetry generator

- $G[\theta]$ generates standard gauge transformations, $\delta A_\mu = -\partial_\mu \theta$, because it is satisfied $\theta_u = -\dot{\theta}$ [Castellani 1974]

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- **New symmetry generator** $S[\eta]$ has an r-independent parameter $\eta_A(y)$ (to eliminate $\chi^A_{(n)}$, $n \geq 1$)
- **Transformation law of the fields**

$$\boxed{\begin{aligned} \delta_\theta A_\mu &= -\partial_\mu \theta\,, & \delta_\eta A_\mu &= \epsilon\, \eta_A\, \delta^A_\mu \\ \delta_\theta \pi^\mu &= 0\,, & \delta_\eta \pi^\mu &= \frac{1}{e^2}\, \delta^\mu_r \sqrt{\gamma}\, \nabla_A \eta^A \end{aligned}}$$

- **Improper transformations**: $\theta_{(0)}$, η^A
  [Benguria, Cordero, Teitelboim 1977]

★ η^A acts on the boundary fields only

$$\delta_\eta A_{(0)A} = \epsilon\, \eta_A\,, \quad \delta_\eta A_{(n)A} = 0\,, \quad n \geq 1 \qquad \text{(similarly for } \pi^r\text{)}$$

A new symmetry generator

Improved generators and charges

- **Improved generators**

$$G_Q[\theta] = G[\theta] + Q[\theta] \qquad (Q[\theta] = \text{ surface term})$$
$$S_Q[\eta] = S[\eta] + Q_s[\eta] \qquad (Q_s[\theta] = \text{ surface term})$$

- **Differentiability**

$$\delta G_Q[\theta] = \int \mathrm{d}^3 x \left(\frac{\delta G_Q[\theta]}{\delta A_\mu} \delta A_\mu + \frac{\delta G_Q[\theta]}{\delta \pi^\mu} \delta \pi^\mu \right)$$
$$\delta S_Q[\eta] = \int \mathrm{d}^3 x \left(\frac{\delta S_Q[\eta]}{\delta A_\mu} \delta A_\mu + \frac{\delta S_Q[\eta]}{\delta \pi^\mu} \delta \pi^\mu \right)$$

A new symmetry generator

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- **Improved generators**

$$G_Q[\theta] = G[\theta] + Q[\theta] \quad (Q[\theta] = \text{ surface term})$$
$$S_Q[\eta] = S[\eta] + Q_s[\eta] \quad (Q_s[\theta] = \text{ surface term})$$

- **Differentiability**

$$\delta G_Q[\theta] = \int \mathrm{d}^3x \left(\frac{\delta G_Q[\theta]}{\delta A_\mu} \delta A_\mu + \frac{\delta G_Q[\theta]}{\delta \pi^\mu} \delta \pi^\mu \right)$$
$$\delta S_Q[\eta] = \int \mathrm{d}^3x \left(\frac{\delta S_Q[\eta]}{\delta A_\mu} \delta A_\mu + \frac{\delta S_Q[\eta]}{\delta \pi^\mu} \delta \pi^\mu \right)$$

- **Charges**

$$Q[\theta] = -\oint \mathrm{d}^2y\, \theta\, \pi^r$$
$$Q_s[\eta] = \frac{1}{e^2} \oint \mathrm{d}^2y \sqrt{\gamma}\, \eta^A A_A$$

- Infinite number of global charges (Laurent coefficients).

A new symmetry generator

Charge algebra

- **Reduced phase space**: $G_Q[\theta] = Q[\theta],\ S_Q[\eta] = Q_s[\eta]$
- **Abelian charge algebra**

$$\boxed{\begin{aligned} \{Q[\theta_1], Q[\theta_2]\} &= 0 \\ \{Q_s[\eta_1], Q_s[\eta_2]\} &= 0 \\ \{Q[\theta], Q_s[\eta]\} &= C[\theta,\eta] \end{aligned}}$$

- **Central charge** $C[\theta,\eta] = \frac{1}{e^2}\oint \mathrm{d}^2y\,\sqrt{\gamma}\,\eta^A\partial_A\theta \neq 0$

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- **Central charge** $C[\theta,\eta] = \frac{1}{e^2}\oint \mathrm{d}^2y\,\sqrt{\gamma}\,\eta^A\partial_A\theta \neq 0$
- Holographic conjugate pairs on $\mathbb{S}^2$ [Donnay,Puhm, Strominger 2019]

$\{Q[\theta], Q_s[\eta]\} = C[\theta,\eta] \quad \leftrightarrow \quad \{q,p\} = 1$

$Q[\theta]$ – conformally soft photon mode

$Q_s[\eta]$ – Goldstone current

A new symmetry generator

Mode expansion of the charge algebra

- **Laurent series**

$$\psi(z,\bar{z}) = \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \frac{\psi_{nm}}{z^{n+h}\bar{z}^{m+\bar{h}}}$$

- The powers $(h,\bar{h})$ are related to the spin of the tensor ψ

- Scalars $\pi^r : (0,0)$
- Vectors $A_z : (1,0)$, $A_{\bar{z}} : (0,1)$

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- **Charges**

$$Q[\theta] = \sum_{n,m} \theta_{nm} G_{nm} \in \mathbb{R}$$

$$Q_s[\eta] = \sum_{n,m} (\eta_{nm}\bar S_{nm} + \bar\eta_{nm} S_{nm}) \in \mathbb{R}$$

- **Generators**

$$G_{nm} = 4\pi^2\, \pi_{1-n,1-m}$$

$$S_{nm} = -\frac{4\pi^2}{e^2} A_{-n,-m}$$

$$\bar S_{nm} = -\frac{4\pi^2}{e^2} \bar A_{-n,-m}$$

A new symmetry generator

- **Algebra** (non vanishing brackets only)

$$\boxed{\begin{aligned} \{G_{nm}, S_{kl}\} &= \kappa n\,\delta_{n+k,0}\delta_{m+l,0} \\ \{G_{nm}, \bar{S}_{kl}\} &= \kappa m\,\delta_{n+k,0}\delta_{m+l,0} \end{aligned}}$$

- **Level of the algebra:** $\kappa = \frac{4\pi^2}{e^2}$

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- **Level of the algebra:** $\kappa = \frac{4\pi^2}{e^2}$
- **Change of the basis**: $(G_{nm}, S_{nm}, \bar{S}_{nm}) \to (R_{nm}, J_{nm}, \bar{J}_{nm})$
- **Generalization of the Kac-Moody algebra**

$$\begin{aligned} \{J_{nm}, J_{kl}\} &= \kappa\,(n-m)\;\delta_{n+k,0}\delta_{m+l,0} \\ \{\bar{J}_{nm}, \bar{J}_{kl}\} &= -\kappa\,(n-m)\;\delta_{n+k,0}\delta_{m+l,0} \\ \{R_{nm}, J_{kl}\} &= \kappa n\, \delta_{n+k,0}\delta_{m+l,0} \\ \{R_{nm}, \bar{J}_{kl}\} &= \kappa\, m\, \delta_{n+k,0}\delta_{m+l,0} \\ \{R_{nm}, R_{kl}\} &= \kappa\,(n+m)\, \delta_{n+k,0}\delta_{m+l,0} \end{aligned}$$

A new symmetry generator

Abelian Kac-Moody subalgebras

- **We obtain six Abelian KM algebras** $\{j_n, j_m\} = \kappa n\, \delta_{n+m,0}$

Currents j_n	Levels
J_{n0}, J_{0n}	$\kappa, -\kappa$
$\bar{J}_{n0}, \bar{J}_{0n}$	$-\kappa, \kappa$
R_{n0}, R_{0n}	κ, κ

- **Non vanishing mixed brackets**: $\{R_{n0}, J_{m0}\}, \{R_{0n}, \bar{J}_{0m}\} \neq 0$
- Each KM algebra is naturally generated by a current that is a holomorphic or anti-holomorphic function.

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- Each KM algebra is naturally generated by a current that is a holomorphic or anti-holomorphic function.
- $\{J_{00}, \bar{J}_{00}, R_{00}\}$ span the global Abelian algebra (only two independent in EM)

A new symmetry generator

Beyond $\mathrm{U}(1)$ – conformal symmetry

- Conformal plane – a realization of conformal symmetry described by Virasoro algebra
- Virasoro algebra – obtained from KM algebra using the Sugawara construction [Sugawara 1967]

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- Quantization introduces a central extension.
- *In progress*: Relation to the global 4D Poincaré generators.

Extension to Yang-Mills theory

Yang-Mills theory

$$I[A] = -\frac{1}{4g^2}\int \mathrm{d}^4x\sqrt{\mathfrak{g}}\,F_a^{\mu\nu}F_{\mu\nu}^a$$

- **Constraints**

$$\pi_a^u \approx 0\,, \quad \chi_a^A \equiv \epsilon\,\pi_a^A - \frac{1}{g^2}\sqrt{\gamma}\gamma^{AB}F_{rB}^a \approx 0, \quad \chi_a \equiv D_i\pi_a^i \approx 0$$

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- **Constraint algebra**

$$\begin{aligned}
\{\chi_a, \chi'_b\} &= f_{ab}{}^{c}\,\chi_c\,\delta^{(3)} \\
\left\{\chi_a, \chi'^A_b\right\} &= f_{ab}{}^{c}\,\chi^A_c\,\delta^{(3)} \\
\left\{\chi^A_a, \chi'^B_b\right\} &= \Omega^{AB}_{ab}(x,x')
\end{aligned}$$

- **Non-Abelian symplectic matrix**

$$\Omega^{AB}_{ab}(x,x') = -\frac{2\epsilon}{g^2}\sqrt{\gamma}\gamma^{AB}\left(g_{ab}\,\partial_r + f_{abc}\,A^c_r\right)\delta^{(3)}$$

Extension to Yang-Mills theory

- **Zero mode**

$$\partial_r V_A = -[A_r, V_A] \quad \Rightarrow \quad V_A(x) = U^{-1} V_{(0)A}(y) U,$$

with $U = \exp\left(-\int_r^\infty \mathrm{d}r\, A_r\right)$ and the bdy. condition $V_A|_{r\to\infty} = V_{(0)A}(y)$

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- **Charges**

$$Q[\theta] = -\oint \mathrm{d}^2 y\, \theta^a \pi_a^r\,, \quad Q_s[\eta] = \frac{1}{g^2}\oint \mathrm{d}^2 y\, \sqrt{\gamma}\, \eta_A^a A_a^A$$

- **Local transformations**

$$\begin{aligned} \delta_{\theta,\eta} A_u^a &= \theta_u^a \qquad \delta_{\theta,\eta} A_r^a = -D_r \theta^a \\ \delta_{\theta,\eta} A_A^a &= -D_A \theta^a + \epsilon\, \eta_A^a \end{aligned}$$

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$$\delta_{\theta,\eta} A^a_u = \theta^a_u \qquad \delta_{\theta,\eta} A^a_r = -D_r \theta^a$$
$$\delta_{\theta,\eta} A^a_A = -D_A \theta^a + \epsilon\, \eta^a_A$$

- **Non-Abelian charge algebra**

$$\boxed{\begin{aligned} \{Q[\theta_1], Q[\theta_2]\} &= Q[[\theta_1, \theta_2]] && \longrightarrow Q_s \text{ is non-Abelian} \\ \{Q[\theta], Q_s[\eta]\} &= Q_s[[\theta, \eta]] + \frac{1}{g^2} \oint \mathrm{d}^2 y\, \sqrt{\gamma}\, \eta^A_a \partial_A \theta^a \\ \{Q_s[\eta_1], Q_s[\eta_2]\} &= 0 && \longrightarrow Q_s \text{ is Abelian} \end{aligned}}$$

Extension to Yang-Mills theory

- **Mode algebra**

$$\boxed{\begin{aligned} \left\{G^a_{nm}, G^b_{kl}\right\} &= f^{ab}_{\ \ c}\, G^c_{n+k,m+l} \\ \left\{G^a_{nm}, S^b_{kl}\right\} &= f^{ab}_{\ \ c}\, S^c_{n+k,m+l} + \kappa n g^{ab} \delta_{n+k,0}\delta_{m+l,0} \\ \left\{G^a_{nm}, \bar{S}^b_{kl}\right\} &= f^{ab}_{\ \ c}\, \bar{S}^c_{n+k,m+l} + \kappa m g^{ab} \delta_{n+k,0}\delta_{m+l,0} \end{aligned}}$$

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- **Level** $\kappa = \frac{4\pi^2}{g^2}$
- One can apply the Sugawara method again...
- *Symmetries at the asymptotic null boundary, described by KM algebras and Virasoro algebras, are general features of 4D gauge theories*

Discussion

Degrees of freedom count

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- Dirac formula $\quad \mathrm{d.o.f.} = N - N_{1^{st}\mathrm{class}} - \frac{1}{2}\, N_{2^{nd}\mathrm{class}}$

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- Electromagnetism

A_μ	$N = 4$
$\pi^u,\ \chi = \partial_i \pi^i$	$N_{1^{st}\text{class}} = 2$
χ_A	$N_{2^{nd}\text{class}} = 2$

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- $\mathrm{d.o.f.} = 4 - 2 - \frac{1}{2}\, 2 = 1$ **WRONG!!** $\mathrm{d.o.f.} = 2$
- The Dirac formula is applicable only when the multipliers are either arbitrary (1^{st} class constraints) or fully determined (2^{nd} class constraints).

\- It fails when the multipliers satisfy a differential equation.

\- In the nul foliation: $\quad \partial_r \lambda^A = f^A \quad \Rightarrow \quad \lambda^A = \Lambda^A(y) + \bar{\lambda}^A$

Discussion

Asymptotic conditions

- Invariance of boundary conditions under Poincaré transformatons is not straighforward
- Hamiltonian treatment at spatial infinity needs additional **parity conditions** to ensure invariance under boosts

- Electromagnetism [Henneaux, Troessaert 2018]
- Yang-Mills [Tanzi, Giulini 2020]

- Null-slices foliated standard b.c. in electromagnetism are invariant under Poincaré group. [Bunster, Gomberoff, Pérez 2018]
- We showed the Poincaré invariance in the non-Abelian case.

Discussion

Poincaré transfromations

- We found several Kac-Moody algebras, but not all of them are related to the global Poincaré symmetry in 4D spacetime.
- We constructed a generator of 4D Poincaré transformations and its action at the light front, by writing the YM stress tensor in the canonical form,

$$T^{\mu}_{\ \nu} = \frac{1}{g^2} \left(F_a^{\mu\alpha} F^a_{\nu\alpha} - \frac{1}{4}\, \delta^{\mu}_{\nu}\, F_a^{\alpha\beta} F^a_{\alpha\beta} \right)$$

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- We are working on showing its relation with the Virasoro generators.

To be done

- Hamiltonian treatment of the gravitatonal action using the null foliation.
- Description of a holographic theory.
- Addition of the θ-term in the action (Pontryagin topological invariant with the couplig θ), which will change the central charges.

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T H A N K   Y O U !

THANK YOU!

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Black holes and asymptotic symmetries